PROBLEMS IN MEASURE THEORY

1. Topology of the real line and plane

Say that two rectangles are *disjoint* if their closures do not intersect, and *almost disjoint* if their interiors do not intersect.

Problem 1.1. *Let U be an open subset of the real line. For $x \in U$, let*

$$a_x = \inf\{a < x : (a, x) \subseteq U\},\ \ b_x = \sup\{b > x : (x, b) \subseteq U\}.$$

If $I_x = (a_x, b_x)$, then verify the following:

(1) $U = \bigcup_{x \in U} I_x$.
(2) $I_x \cap I_y \neq \emptyset$ *if and only if* $I_x = I_y$.

Conclude that every open subset of the real line can be written as a countable union of disjoint open intervals.

Problem 1.2. *Can an open disc in the plane be written as a countable union of disjoint open rectangles?*

Hint. Recall the definition of disjoint open rectangles.

Problem 1.3. *Let U be an open set in the plane. Follow the following instructions:*

(1) *Consider the grid in the plane formed by taking all closed squares S of side length 1 whose vertices have integer coefficients. Here are some guide lines:*
    Accept S if $S \subseteq U$,
    Reject S if $S \subseteq$ complement of U,
    Tentatively accept S otherwise.
(2) *As a second step, bisect the tentatively accepted squares of length 1 into 4 squares with side length $1/2$. Again follow the guide lines given in (1).*
(3) *At each step, bisect the tentatively accepted squares of length l into 4 squares with side length $l/2$. Again follow the guide lines given in (1).*
(4) *Repeat the above procedure indefinitely to obtain a countable collection of almost disjoint open squares.*

Conclude that any open subset of the plane can be written as a countable union of almost disjoint open squares.

2. Naive Measure Theory

Problem 2.1. *If a rectangle R is union of finitely many rectangles $R_1, \cdots, R_N$ then show that*

$$Area(R) \leq \sum_{i=1}^{N} Area(R_k),$$

where the equality holds if the union is disjoint.

Hint. This is not obvious. To get an idea, draw a picture for $N = 3$.

Problem 2.2. *Let C denote the Cantor set. Show that for $\epsilon > 0$, there exist finitely many intervals $I_1, \cdots, I_k$ in the real line such that*

$$C \subseteq \bigcup_{i=1}^{k} I_i \text{ and } \sum_{i=1}^{k} \text{length of } I_i < \epsilon.$$

Problem 2.3. *Let L be a finite line segment in the plane. Show that for $\epsilon > 0$, there exist finitely many squares $S_1, \cdots, S_k$ in the plane such that*

$$L \subseteq \bigcup_{i=1}^{k} S_i \text{ and } \sum_{i=1}^{k} \text{area of } S_i < \epsilon.$$

Whether similar assertion holds true for a line in the plane?

Let U be an open subset of the plane. By Problem 1.3, U may be written as a countable union of disjoint open squares S_n. We may define the "area" of U to be the sum of the area of S_n. The fact that this definition is independent of the choice of S_n lies deeper.

Problem 2.4. *If U is an open subset of the plane then for any x in the plane, show that the area of the coset $U + x = \{u + x : u \in U\}$ of U is same as that of U.*

Problem 2.5. *Let U be an open square in the plane. Show that the area of U is less than or equal to*

$$\inf \sum_{j=1}^{\infty} Area(Q_j),$$

where the infimum is taken over all countable coverings $U \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed squares.

Hint. Use the compactness of $\overline{U}$ and then apply Problem 2.1.

Problem 2.6. *Let L denote a line in the plane. Show that*

$$\inf \sum_{j=1}^{\infty} Area(Q_j) = 0$$

where the infimum is taken over all countable coverings $L \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed squares.

Problem 2.7. *Construct a Cantor-like set in the plane.*

3. Lebesgue Measurable Sets

Problem 3.1. *Let A be a subset of $[0, 1]$ consisting of all numbers which do not have the digit 4 appearing in their decimal expansion. Find the Lebesgue measure of A.*

Problem 3.2. *Consider the set D obtained by taking the union of open intervals which are deleted at the steps $1, 3, 5, \cdots$ in the construction of a Cantor set. Show that the boundary of D has positive Lebesgue measure.*

Problem 3.3. *Show that the Lebesgue measure of any line in the plane is zero.*

Problem 3.4. *Find the Lebesgue measure of the set constructed in Problem 2.7.*

Problem 3.5. *Let A be a bounded subset of the real line of positive Lebesgue measure and let $D = \{x - y : x, y \in A\}$.*

(1) *Verify that the function g given by*

$$g(t) = \text{ Lebesgue measure of } A \cap (A + t) \text{ for real number } t$$

is continuous, where $A + t = \{x + t : x \in A\}$.

(2) *Use (1) to prove that D contains an interval.*

References

[1] E. Stein and R. Shakarchi, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, Priceton and Oxford, 2005.